12/9/24

MATH4030 Tutorial

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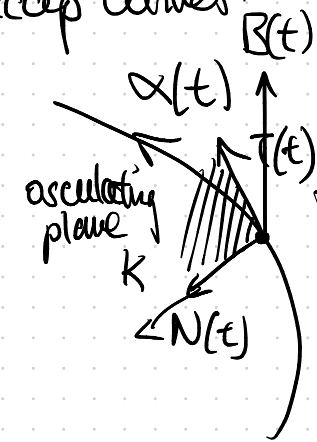
LSB 232A

Objectives:

- Review curves
- Practice problems related to curvature, torsion, Frenet Frame.

Recap Curves:



arc-length param. s s.t. $|\alpha'(s)| = 1$. $s(t) = \int_0^t |\alpha'(u)|\,du$

Frenet Frame $\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & K & 0 \\ -K & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$

$B'(s) = -\tau(s) N(s)$

Lecture notes

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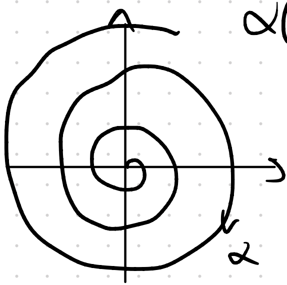
$B'(s) = \tau(s) N(s)$.

Non-arc-length param.

$$K(t) = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3} \qquad \tau(t) = \frac{\langle \alpha'(t) \times \alpha''(t), \alpha'''(t) \rangle}{|\alpha'(t) \times \alpha''(t)|^2}$$

Q1: Compute the arc-length, curvature, torsion of the Logarithmic Spiral given by param.

$$\alpha(t) = (ae^{bt}\cos t,\ ae^{bt}\sin t,\ 0) \qquad a>0,\ b<0 \text{ constants}$$



Sol'n: α is plane curve, so $\tau \equiv 0$.

arc-length: $s(t) = \int_0^t |\alpha'(u)|\,du.$

$$\alpha'(u) = (abe^{bu}\cos u - ae^{bu}\sin u,\ abe^{bu}\sin u + ae^{bu}\cos u,\ 0)$$

$$|\alpha'(u)|^2 = a^2b^2e^{2bu}\cos^2 u + a^2e^{2bu}\sin^2 u + a^2b^2e^{2bu}\sin^2 u + a^2e^{2bu}\cos^2 u$$

$$= a^2e^{2bu}(1+b^2)$$

$$|\alpha'(u)| = ae^{bu}\sqrt{1+b^2}$$

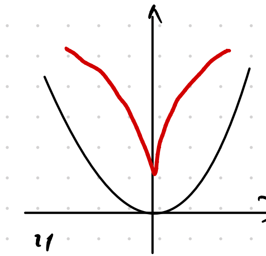
$$s(t) = \int_0^t ae^{bu}\sqrt{1+b^2}\,du = \frac{a}{b}\sqrt{1+b^2}\,(e^{bt}-1)$$

$$\kappa(t) = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3}$$

$$\alpha''(t) = (ab^2e^{bt}\cos t - 2abe^{bt}\sin t - abe^{bt}\cos t,\ ab^2e^{bt}\sin t + 2abe^{bt}\cos t - ae^{bt}\sin t,\ 0)$$

$$\alpha'(t) \times \alpha''(t) = (0, 0, a^2(1+b^2)e^{2bt})$$

$$\kappa(t) = \frac{a^2(1+b^2)e^{2bt}}{a^3e^{3bt}(1+b^2)^{3/2}} = \frac{1}{ae^{bt}\sqrt{1+b^2}}$$



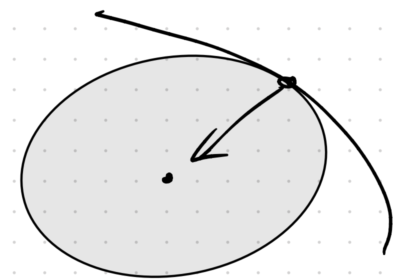
Q2: $\alpha : I \to \mathbb{R}^2$ regular plane curve w/ $K(t) \neq 0$ $\forall t \in I$.

$$\beta(t) = \alpha(t) + \frac{1}{K(t)} N(t), \quad t \in I.$$

is called the evolute of α. "locus of centres of curvature of α"

1) Show that the tangent at t of the evolute β is the normal to α at t.

2) Consider normal lines of α at two points t_1, t_2, $t_1 \neq t_2$. Let t_1 approach t_2 and show that the intersection points of the normals converge to a point on the evolute.



Sol'n: 1) WLOG let s be arc-length param.

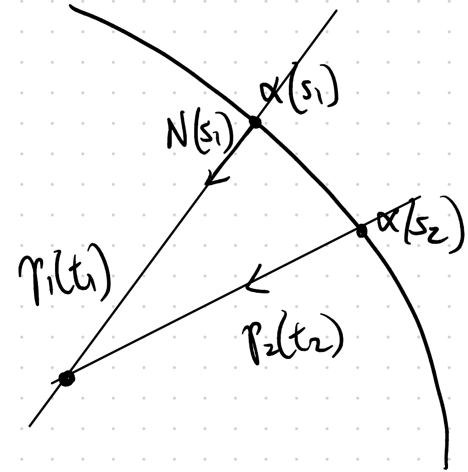
$$\beta'(s) = \alpha'(s) + \left(\frac{1}{K(s)} N(s)\right)'$$

$N'(s) = K(s)T(s) - \tau(s)B(s)$

$$= \alpha'(s) + \frac{1}{K(s)} N'(s) - \frac{K'(s)}{K^2(s)} N(s)$$

$$= \cancel{T(s)} - \cancel{\frac{K(s)}{K(s)} T(s)} - \frac{\tau(s)}{K(s)} B(s) - \frac{K'(s)}{K^2(s)} N(s)$$

$$= \frac{-\kappa'(s)}{\kappa^2(s)} N(s).$$



2) Normal lies at α: WTS as $s_2 \to s_1$, $t_1 \to \frac{1}{\kappa(s_1)}$.

$$\gamma_1(t_1) = \alpha(s_1) + t_1 N(s_1)$$

$$\gamma_2(t_2) = \alpha(s_2) + t_2 N(s_2)$$

Point of intersection given by t_1, t_2 s.t. $\alpha(s_1) + t_1 N(s_1) = \alpha(s_2) + t_2 N(s_2)$

$$\Rightarrow \frac{\alpha(s_1) - \alpha(s_2)}{s_2 - s_1} = \frac{t_2 N(s_2) - t_1 N(s_1)}{s_2 - s_1}$$

Take inner product w/ $T(s_1)$, then LHS $= \left\langle T(s_1), \lim_{s_2 \to s_1} \frac{\alpha(s_2) - \alpha(s_1)}{s_2 - s_1} \right\rangle$

$$= |T(s_1)|^2 = 1$$

Take limit $s_2 \to s_1$, then $t_2 \to t_1$ and RHS becomes $-\lim_{s_2 \to s_1} t_1 \lim_{s_2 \to s_1} \frac{N(s_2) - N(s_1)}{s_2 - s_1}$

$$= -\lim_{s_2 \to s_1} t_1 \, N'(s_1).$$

Then taking inner product w/ $T(s_1)$, we get

$$\text{RHS} \to -\lim_{s_2 \to s_1} t_1 \langle T(s_1), N'(s_1) \rangle$$

$$= -\lim_{s_2 \to s_1} t_1 \langle T(s_1), -\kappa(s_1) T(s_1) - \tau(s_1) B(s_1) \rangle$$

$$= -\lim_{s_2 \to s_1} t_1 \; -\kappa(s_1).$$

So we have $\lim_{s_2 \to s_1} t_1 = \frac{1}{\kappa(s_1)}$ as required. //.

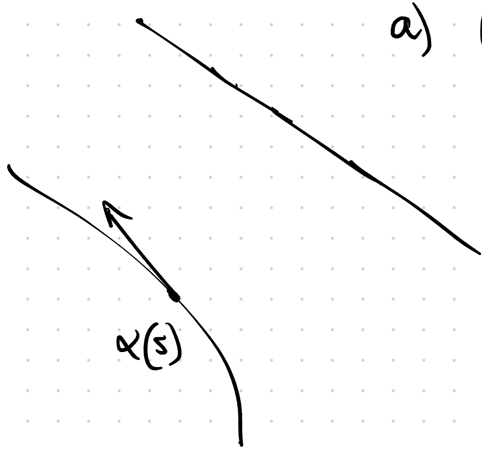
Q3: a regular param. curve α has all its tangent lines passing through a fixed point.

Show that

a) The trace of α is a (segment of a) straight line

b) Does the conclusion hold when α is not regular?

Soln:



a) Let $x_0 \in \mathbb{R}^3$ be the fixed point.

The condition means that $\forall s \in \mathbb{R}$, $\exists\, \lambda(s)$ s.t.

$$\alpha(s) + \lambda(s)\alpha'(s) = x_0$$

Differentiating wrt s, we have

$$\alpha'(s) + \lambda'(s)\alpha'(s) + \lambda(s)\alpha''(s) = 0.$$

$$\Rightarrow (1+\lambda')\,\alpha'(s) + \lambda(s)\,\alpha''(s) = 0. \quad (1)$$

By regular condition, we can take $|\alpha'(s)| = 1$ $\Leftarrow$ differentiating this gives

$\langle \alpha'(s), \alpha''(s) \rangle = 0$.

Then taking inner product with $\alpha''(s)$ in (1), we get

$\lambda(s) |\alpha''(s)| = 0 \Rightarrow$ Case 1: if $\alpha''(s) \neq 0$, then $\lambda(s) = 0$ and $\alpha(s) = x_0$ at that x_0.

Case 2: $\alpha''(s) = 0$. Then we automatically have $\kappa(s) = 0$ at that s, since $\kappa(s) = |\alpha''(s)|$ and so α is straight.

b) Consider $\alpha(t) = \begin{cases} (t^3, 0, 0) & t \in [-1, 0] \\ (0, t^3, 0) & t \in [0, 1]. \end{cases}$

curve is C^2, not regular at 0. so α'' is well-defined.

but all tangents pass through origin.